Department of Applied Physics
Entrance Examination Booklet

Physics

(Answer the 4 Problems in this Booklet)

August 27 (Tuesday) 9:00 – 13:00, 2024

REMARKS

1. Do not open this booklet before the start is announced.
2. Inform the staff when you find misprints in the booklet.
3. Answer the four problems in this booklet.
4. Use one answer sheet for each problem (four answer sheets are given). You may use the back side of each answer sheet if necessary.
5. Write down the number of the problem which you answer in the given space at the top of the corresponding answer sheet.
6. You may use the draft sheets of this booklet to make notes, but you must not detach them.
7. Any answer sheet with marks or symbols irrelevant to your answers will be considered invalid.
8. Do not take this booklet and the answer sheets with you after the examination.

Examinee number	No.

Write down your examinee number above

Problem 1

Consider the one-dimensional motion of a particle connected to one end of a spring. The particle has mass m and the spring constant is $m\omega_0^2$ ($\omega_0 > 0$). The other end of the spring is attached to an immobile wall. The position $x(t)$ of the particle is zero when the spring is of natural length. The particle is subject to a friction force $-\zeta \frac{\mathrm{d}x(t)}{\mathrm{d}t}$ and an external force $f(t)$. Here, $\zeta > 0$ is the friction coefficient and t is time. In the following questions, i represents the imaginary number unit.

[1] First, consider the case where the external force $f(t)$ is a periodically oscillating force. Here, the position $x(t)$ and the external force $f(t)$ are expressed as complex numbers. In particular, let $f(t)$ be defined as $f(t) = F_0 e^{i\Omega t}$ with an angular frequency $\Omega > 0$.

[1.1] Find $x(t)$ in steady state.

[1.2] Defining the response function $\varepsilon = x(t)/f(t)$, find the Ω dependences of the real part ε' and the imaginary part ε'' of ε.

[1.3] When $\zeta \ll m\omega_0$ and $|\Omega - \omega_0| \ll \omega_0$, draw the approximate shape of the Ω dependences of ε' and $-\varepsilon''$. In particular, annotate the angular frequencies where ε' and $-\varepsilon''$ reach their extrema.

[2] Next, consider the case where the external force $f(t)$ is a fluctuating force. Here, the position $x(t)$ and the external force $f(t)$ are expressed as real numbers. Let $f(t)$ be represented by a random function satisfying $\langle f(t)f(t+\tau)\rangle = I\delta(\tau)$ and $\langle f(t)\rangle = 0$. I is a positive constant. For some physical quantity a, $\langle a\rangle$ represents the ensemble mean (the mean in an ensemble of countless identical systems). $\delta(\tau)$ is the Dirac delta function. In the following questions, we consider the steady state of the motion of the particle when $\zeta < 2m\omega_0$.

[2.1] When $\left\langle \left(\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right)^2 \right\rangle$ is represented as $\overline{v^2}$, namely, $\left\langle \left(\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right)^2 \right\rangle = \overline{v^2}$, express the average squared displacement of the particle $\langle x(t)^2\rangle$ in terms of $\overline{v^2}$. Also, show the derivation of this result. You may use the fact that there is no correlation between $f(t)$ and $x(t)$: $\langle f(t)x(t)\rangle = \langle f(t)\rangle\langle x(t)\rangle$.

[2.2] Express the Fourier transform $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}\,\mathrm{d}t$ of $x(t)$ in terms of the Fourier transform $F(\omega)$ of $f(t)$. In addition, express the power spectrum $I_x(\omega) = \langle |X(\omega)|^2\rangle$ in terms of the power spectrum $I_f(\omega) = \langle |F(\omega)|^2\rangle$ of $f(t)$.

[2.3] Find the correlation function $\langle x(t)x(t+\tau)\rangle$ of $x(t)$, using the general relation $I_a(\omega) = \int_{-\infty}^{\infty} \langle a(t)a(t+\tau)\rangle e^{-i\omega\tau}\,\mathrm{d}\tau$ between the power spectrum $I_a(\omega)$ of the physical quantity $a(t)$ and the correlation function $\langle a(t)a(t+\tau)\rangle$ of $a(t)$. Here, $a(t)$ is a real-valued function. You may define $\omega_1 = \sqrt{\omega_0^2 - \frac{\zeta^2}{4m^2}}$ and include it in your answer.

[2.4] Express I in terms of $\overline{v^2}$, by setting $\tau = 0$ in the expression for the correlation function obtained in Question [2.3] and then comparing to the result of Question [2.1].

Problem 2

Consider quantum mechanical particles scattered by a potential barrier in one dimension. The time-independent Schrödinger equation is given by

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + V(x)\psi(x) = \mathcal{E}\psi(x), \tag{1}$$

where m is the mass of a particle, $\hbar$ is Planck's constant divided by 2π, $V(x)$ is the potential, and $\mathcal{E}$ is the energy of the particle. Here, we consider the case where particles propagating in the positive direction of the x-axis hit the potential barrier. Answer the following questions with i as the imaginary number unit.

[1] Let us consider a potential barrier given by $V(x) = V_0$ ($|x| \le a$) and $V(x) = 0$ ($|x| > a$). When $0 < \mathcal{E} < V_0$, the wave function of the particle is written as

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & (x < -a) \\ Ce^{\gamma x} + De^{-\gamma x} & (|x| \le a) \\ Ee^{ikx} & (x > a), \end{cases} \tag{2}$$

where $k = \sqrt{\frac{2m}{\hbar^2}\mathcal{E}}$, $\gamma = \sqrt{\frac{2m}{\hbar^2}(V_0 - \mathcal{E})}$, and V_0 and a are positive constants. Answer Questions [1.1] to [1.5] using k and γ but not $\mathcal{E}$ or V_0.

[1.1] Using the boundary conditions that the wave function and its derivative function are continuous, find two equations relating the coefficients A, B, C and D.

[1.2] Similarly, find two equations relating the coefficients C, D and E.

[1.3] From the equations derived in Questions [1.1] and [1.2], we obtain

$$\frac{B}{E} = \left(\frac{ik-\gamma}{2ik}\right)\left(\frac{ik+\gamma}{2\gamma}\right)e^{-2\gamma a} - \left(\frac{ik+\gamma}{2ik}\right)\left(\frac{ik-\gamma}{2\gamma}\right)e^{2\gamma a}. \tag{3}$$

Using this relation, find the transmission probability (transmittance) T_1 of the particle passing through the potential barrier. You may use the hyperbolic functions $\cosh(\alpha) = \frac{e^{\alpha} + e^{-\alpha}}{2}$ and $\sinh(\alpha) = \frac{e^{\alpha} - e^{-\alpha}}{2}$.

[1.4] Express T_1 in terms of k and a when $\frac{\gamma}{k} \ll 1$ and $a\gamma \ll 1$.

[1.5] Express T_1 in terms of k, γ and a when $a\gamma \gg 1$.

[2] Let us consider the case with $\mathcal{E} > V_0$ for the same potential barrier as in Question [1]. Answer Questions [2.1] and [2.2] using $\mathcal{E}$ and V_0 but not k or γ.

[2.1] Find the $\mathcal{E}$ dependence of the transmittance T_2 in this case.

[2.2] Plot schematically the $\mathcal{E}$ dependence of T_2 with $\frac{\mathcal{E}}{V_0}$ as the variable on the horizontal axis, noting the value of T_2 when $\mathcal{E} \gg V_0$. Also, find all conditions where T_2 is at a local maximum.

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[3] Next, we consider particles scattered by a potential barrier given by $V(x) = 0$ $(x < -a)$, $V(x) = V_0$ $(|x| \leq a)$ and $V(x) = -V_1$ $(x > a)$, where $V_1 > 0$ and $0 < \mathcal{E} < V_0$.

[3.1] Express the transmittance T_3 and the reflectance R_3 using k, γ and k_1 but not $\mathcal{E}, V_0$ or V_1. Here, we define $k = \sqrt{\frac{2m}{\hbar^2}\mathcal{E}}$, $\gamma = \sqrt{\frac{2m}{\hbar^2}(V_0 - \mathcal{E})}$, and $k_1 = \sqrt{\frac{2m}{\hbar^2}(\mathcal{E} + V_1)}$.

[3.2] Find the asymptotic value of T_3 when increasing V_1 while keeping $\mathcal{E}$ fixed.

Problem 3

We consider the Ising model, where N spins are each interacting with all the other spins with a uniform coupling constant. Here, N is a natural number. The energy is given by

$$E = -\frac{J}{N}\sum_{\alpha<\beta}\sigma_\alpha\sigma_\beta, \tag{1}$$

where σ_α and σ_β are variables taking values of ± 1, and J is a nonzero real number. We assume that α and β represent natural numbers satisfying $1 \le \alpha \le N$ and $1 \le \beta \le N$. The sum is taken for all combinations satisfying the condition $\alpha < \beta$.

[1] First, we consider the cases with $N = 2$ and 3.

[1.1] Equation (1) for $N = 2$ is written as

$$E = -\frac{J}{2}\sigma_1\sigma_2. \tag{2}$$

Determine all the energies E and the corresponding degeneracies W.

[1.2] Equation (1) for $N = 3$ is written as

$$E = -\frac{J}{3}\left(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3\right). \tag{3}$$

Determine all the energies E and the corresponding degeneracies W.

[1.3] Considering the cases of positive J and negative J separately, determine the degeneracy W of the ground state for $N = 3$.

[2] Next, we consider the case where N is an arbitrary natural number. We define the total spin by

$$\sum_{\alpha=1}^{N}\sigma_\alpha = N - 2M, \tag{4}$$

where M is the number of spins taking the value $\sigma_\alpha = -1$.

[2.1] Determine E in terms of M.

[2.2] Considering the cases of positive J and negative J separately, determine the ground state energy E and the corresponding degeneracy W.

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Next, we assume $N \gg 1$ and $J > 0$. Let $m = M/N$ be the fraction of spins with $\sigma_\alpha = -1$ and consider a thermal equilibrium state with the temperature T. Let S be its entropy and C be its heat capacity. Let k_B be the Boltzmann constant.

[3] We consider the thermal equilibrium state with $0 < m < 1/2$.

[3.1] Determine E/N as a function of m.

[3.2] Determine S/N as a function of m. Use the Stirling formula

$$\log n! \simeq n \log n - n, \tag{5}$$

which is a good approximation for sufficiently large n.

[3.3] Determine the temperature T as a function of m using the results of Questions [3.1] and [3.2], and the formula

$$\frac{\mathrm{d}S}{\mathrm{d}E} = \frac{1}{T}. \tag{6}$$

[3.4] In the low temperature limit $k_B T \ll J$, the condition $m \ll 1$ is satisfied. Determine the temperature dependence of m.

[3.5] Using the result of Question [3.4], determine the temperature dependence of S/N in the low temperature limit.

[3.6] Using the result of Question [3.5], determine the temperature dependence of C/N in the low temperature limit.

[4] There is a critical temperature T_c above which $m = 1/2$ in thermal equilibrium. Determine the temperature T_c by taking the limit $m \to 1/2$ in Eq. (6).

Problem 4

Suppose there is an electrically neutral and uniform plasma in which gas atoms are separated into electrons and positive ions. Consider an electromagnetic wave of angular frequency $\omega(> 0)$, which travels in the positive direction of the z-axis in this plasma. It is assumed that only electrons are moved by the electromagnetic field and that the movement of positive ions is negligible. We denote the mass of an electron by m, the charge of an electron by $-e$, and the number density of electrons in the plasma by N.

Note that the Coulomb interaction between particles, collisions between particles, and the effect of gravity need not be considered. Assume that the thermal motion of the electrons is isotropic and has no effect on the electric current density. Assume that the speed of electrons is sufficiently small compared to the speed of light c in vacuum, and that the force exerted on electrons by the magnetic field of electromagnetic waves is negligible. The electric field $\boldsymbol{E}$ of the electromagnetic wave, the velocity $\boldsymbol{v}$ of electrons, and the electric current density $\boldsymbol{J}$ are treated as having only x and y components and as being uniform in the plane perpendicular to the z-axis.

Under these assumptions, we consider the following wave equation for the electric field $\boldsymbol{E}(z,t)$ of the electromagnetic wave at time t,

$$\frac{\partial^2 \boldsymbol{E}(z,t)}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}(z,t)}{\partial t^2} = \frac{1}{c^2 \varepsilon_0}\frac{\partial \boldsymbol{J}(z,t)}{\partial t}. \tag{1}$$

Here, ε_0 is the permittivity of vacuum.

Let us express the electric field of an electromagnetic wave obeying this wave equation in the complex number representation as

$$\boldsymbol{E}(z,t) = \boldsymbol{E}_0 \exp[i(kz - \omega t)], \tag{2}$$

where $\boldsymbol{E}_0$ is a constant vector, i is the imaginary number unit, and k is a complex wavenumber. In answering the questions, you can use the plasma frequency, $\omega_\mathrm{p} = \sqrt{\dfrac{e^2 N}{\varepsilon_0 m}}$, as appropriate.

[1] Write the equation of motion for an electron using the velocity $\boldsymbol{v}(z,t)$ of electrons and the electric field $\boldsymbol{E}(z,t)$ of the electromagnetic wave.

[2] Considering the answer to Question [1], express the time derivative of the current density, $\dfrac{\partial \boldsymbol{J}(z,t)}{\partial t}$, in terms of the electric field $\boldsymbol{E}(z,t)$ of the electromagnetic wave.

[3] Express k^2 as a function of the angular frequency ω. Also find the condition for the angular frequency ω so that the electromagnetic wave can travel for a long distance in this plasma.

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A uniform static magnetic field with magnetic flux density $B_z (> 0)$ is applied to the plasma along the positive direction of the z-axis. Consider an electromagnetic wave with angular frequency ω, which travels in the positive direction of the z-axis in this plasma. Assume that the motion of electrons is caused only by the electric field of the electromagnetic wave and by the static magnetic field. We consider that the electric field $\boldsymbol{E}$ of the electromagnetic wave, the velocity $\boldsymbol{v}$ of electrons, and the current density $\boldsymbol{J}$ all have only x and y components. We express the electric field of the electromagnetic wave as in Eq. (2). In the following, it is assumed that the angular frequency ω of the electromagnetic wave and the plasma frequency ω_{p} are both larger than the cyclotron frequency $\omega_{\mathrm{c}} = \dfrac{eB_z}{m}$.

[4] The x and y components for the electric field $\boldsymbol{E}(z,t)$ of the electromagnetic wave and the velocity $\boldsymbol{v}(z,t)$ of electrons are written as

$$\boldsymbol{E}(z,t) = \begin{pmatrix} E_x(z,t) \\ E_y(z,t) \end{pmatrix}, \quad \boldsymbol{v}(z,t) = \begin{pmatrix} v_x(z,t) \\ v_y(z,t) \end{pmatrix}. \tag{3}$$

Write down the x and y components of the equation of motion for the electron separately.

[5] Assume the current density $\boldsymbol{J}(z,t)$ to be proportional to $\exp[i(kz - \omega t)]$. Then we express the current density in terms of a 2-by-2 matrix $\boldsymbol{\sigma}$ as follows:

$$\boldsymbol{J}(z,t) = \begin{pmatrix} J_x(z,t) \\ J_y(z,t) \end{pmatrix} = \boldsymbol{\sigma} \begin{pmatrix} E_x(z,t) \\ E_y(z,t) \end{pmatrix}. \tag{4}$$

Write down the matrix $\boldsymbol{\sigma}$ in terms of $\varepsilon_0, \omega, \omega_{\mathrm{c}}$, and ω_{p}.

[6] From Eq. (1), we can derive an eigenequation described as

$$\boldsymbol{A} \begin{pmatrix} E_x(z,t) \\ E_y(z,t) \end{pmatrix} = k^2 \begin{pmatrix} E_x(z,t) \\ E_y(z,t) \end{pmatrix}. \tag{5}$$

Here $\boldsymbol{A}$ is a 2-by-2 matrix. When writing the matrix $\boldsymbol{A}$ as

$$\boldsymbol{A} = \left(\frac{\omega}{c}\right)^2 \left[(1-\alpha) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \beta \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right], \tag{6}$$

express α and β in terms of $\omega, \omega_{\mathrm{c}}$, and ω_{p}.

[7] Find all the eigenvalues k^2 and the corresponding eigenvectors of Eq. (5). Here, express k^2 in terms of $c, \omega, \omega_{\mathrm{c}}$, and ω_{p}. Also find the condition for the angular frequency ω so that there are two eigenmodes that can travel for a long distance in the positive direction of the z-axis in this plasma.

[8] Consider the case where the angular frequency ω satisfies the condition derived in Question [7]. Let k_+ and k_- denote the wavenumbers of the two eigenmodes assuming $k_+ > k_- > 0$. Assuming that the electric field of the electromagnetic wave at position $z = 0$ is expressed as

$$\boldsymbol{E}(0,t) = E_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp(-i\omega t), \tag{7}$$

where E_0 is a constant, express the electric field $\boldsymbol{E}(z,t)$ at position $z (> 0)$ in terms of k_+ and k_-. Also explain how the polarization state of this electromagnetic wave changes as it travels in the positive direction of the z-axis.